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EXISTENCE OF SOLUTIONS FOR A NONLINEAR SYSTEM RELATED TO PHOSPHORUS CONCENTRATION IN SHALLOW LAKES

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Abstract

We study, in this work, a system of differential equations related to phosphorus concentration in shallow lakes. This is a nonlinear system of reaction diffusion equations with moving boundary. We obtain the existence of solutions of the system by establishing a prori estimates.

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1. Introduction

Consider in a lake where the water contains certain amount of phosphorus carried in by polluted upstream water or released from the residuals of dead benthos. Microorganism (like algae) eat phosphorus and die. In the process of decaying the dead organism, large amount of oxygen in the water is used. This process is called the *eutriphication*. It is obvious that fast eutriphication in a lake or a reservoir will affect the water quality. Eutriphication control has gained more and more attention recently. Contamination generated from surrounding industrial development and other human activities introduces large amount of nutrients into the water and thus, accelerates the eutrophication process and affect the public water resources. It has been found that phosphorus is the indicator of eutrophication. It is therefore crucial to monitor and control the level of phosphorus in reservoirs or lakes. Very recently, the effect of phosphorus concentration on reservoirs and other water bodies have been studied in [11], [12].

In this work, we study the concentration of phosphates in shallow lakes or reservoirs by considering different factors that affect the phosphorus concentration. The degree of pollution of the water depends on the concentration of phosphorus in the polluted water flowing in, the concentration of phosphate in the sediment at the bottom, the boundary of the water body, and many other factors. With the knowledge of the roles of these factors, we can better control and predicate the quality of the water.

Let $Q \in R_+^3 = \{(x, y, t) : (x, y) \in R^2, t > 0\}$ be a smooth region with a lateral surface $\sum : p(x, y, t) = 0,$ a top $\Omega_T : Q \bigcap \{t = T\}$ and a bottom $\Omega_0 : Q \bigcap \{t = 0\}.$ Denote $c_1(x, y, t)$ as the concentration of phosphate in the lake at location (x, y) and time t, $c_2(x, y, t)$ as the concentration of phosphate at the top layer of the lake bottom at (x, y, t), $c_3(x, y, t)$ as the concentration of algae that absorbs phosphate. We also denote $u(x, y)$

as the speed of water flowing in the x direction, $v(x, y)$ as the speed of water flowing in the y direction, v_s as the rate at which phosphate sinks, v_r as the rate at which the lake bottom releases phosphate, v_b as the rate at which the top layer of lake bottom gets buried by new dirty in the water and newly died micro-plants, and v_p as the rate at which the micro-plants sink. Here v_s, v_r, v_b, v_p are assumed to be independent of (x, y). Then c_1, c_2, c_3 satisfy the following reaction diffusion equations:

$$\frac{\partial c_1}{\partial t} - D_1 \nabla c_1 + u\frac{\partial c_1}{\partial x} + v\frac{\partial c_1}{\partial y} + \frac{v_s}{h}c_1 - \frac{v_r}{h_1}c_2 + c_g \alpha_{pc} c_3 - c_d \alpha pc c_3 = f,$$

$$\frac{\partial c_2}{\partial t} - D_2 \nabla c_2 + \frac{v_b}{h_1}c_2 + \frac{v_r}{h_1}c_2 - \frac{v_s}{h}c_1 - \frac{v_0}{h_1}\alpha_{pc} c_3 = 0,$$

$$\frac{\partial c_3 \alpha_{pc}}{\partial t} - c_g \alpha_{pc} c_3 + c_d \alpha_{pc} c_3 + \frac{v_0}{h_1}\alpha_{pc} c_3 = 0, \tag{1.1}$$

subject to boundary conditions

$$c_1|_{p(x, y, t)=0} = \nu_1(x, y, t)|_{p(x, y, t)=0}, \tag{1.2}$$

$$c_2|_{p(x, y, t)=0} = \nu_2(x, y, t)|_{p(x, y, t)=0}, \tag{1.3}$$

and initial conditions

$$c_1(x, y, 0) = c_{10}(x, y),\ c_2(x, y, 0) = c_{20}(x, y),\ c_3(x, y, 0) = c_{30}(x, y).$$

Here c_{10}, c_{20}, c_{30} are initial densities, ν_1, ν_2 are known functions, $c_g = \nu_m \dfrac{c_1}{kc_2 + c_1}$ is the growth rate for some positive constants ν_m, k, c_d is the death rate of microorganism, α_{pc} is the phosphate/carbon ratio in the water, h_1 denotes the depth of mud containing phosphate at the bottom of lake, h denotes the average depth of the lake, and v_0 is the rate at which the organism sinks.

Consider a related linear system in c_1^*, c_2^*

$$\frac{\partial c_1^*}{\partial t} - D_1 \nabla c_1^* + u\frac{\partial c_1^*}{\partial x} + v\frac{\partial c_1^*}{\partial y} + \frac{v_s}{h} c_1^* - \frac{v_r}{h_1} c_2^* = 0,$$

$$\frac{\partial c_2^*}{\partial t} - D_2 \nabla c_2^* + \frac{v_b}{h_1} c_2^* + \frac{v_r}{h_1} c_2^* - \frac{v_s}{h} c_1^* = 0,$$

subject to boundary conditions

$$c_1^*|_{p(x,y,t)=0} = \nu_1(x, y, t)|_{p(x,y,t)=0}, \tag{1.4}$$

$$c_2^*|_{p(x,y,t)=0} = \nu_2(x, y, t)|_{p(x,y,t)=0}, \tag{1.5}$$

and initial conditions

$$c_1^*(x, y, 0) = 0, \; c_2^*(x, y, 0) = 0.$$

We can homogenize the boundary conditions (1.2), (1.3) by letting $w_1 = c_1 - c_1^*$, $w_2 = c_2 - c_2^*$, $w_3 = c_3$. Therefore, we can assume that $\nu_1(x, y, t) = \nu_2(x, y, t) = 0$.

2. A Vector Operator of Variations

Following Lions' notations [6], we introduce a vector operator. First, denote a bounded open set in R^n as D. Denote the number of derivatives of u (with respect to x) with order less than or equal to $m-1$ as N_1, and the number of derivatives of u (with respect to x) with order m as N_2.

Define a family of functions $A_\alpha^i(x, \eta_1, \cdots, \eta_l; \xi_1, \cdots, \xi_l)$ on $D \times R^{N_1} \times \ldots \times R^{N_1} \times R^{N_2} \times \ldots \times R^{N_2}$, $i = 1, 2, \cdots, l$, with the following properties:

(1) $\forall x \in D$, $A_\alpha^i(x, \eta_1, \cdots, \eta_l; \xi_1, \cdots, \xi_l)$ are continuous on $D \times R^{N_1} \times \ldots \times R^{N_1} \times R^{N_2} \times \ldots \times R^{N_2}$, $i = 1, 2, \cdots, l$;

(2) $\forall(\eta_1, \cdots, \eta_l; \xi_1, \cdots, \xi_l) \in R^{N_1} \times \ldots \times R^{N_1} \times R^{N_2} \times \ldots \times R^{N_2}$, $A_\alpha^i(x, \eta_1, \cdots, \eta_l; \xi_1, \cdots, \xi_l)$ is measurable in x;

(3) There exists a function $k(x) \in L^{p'}(D)$ and a constant C such that, for all $i = 1, 2, \cdots, l$,

$$|A_\alpha^i| \le C(|\eta_1|^{p-1} + \cdots + |\eta_l|^{p-1} + |\xi_1|^{p-1} + \cdots + |\xi_l|^{p-1} + k(x)).$$

Let $D^k u = \{D^\beta : |\beta| = k\}$, $\delta u = \{u, Du, \cdots, D^{m-1}u\}$, $A_\alpha(x, \eta, \xi) = (A_\alpha^1, A_\alpha^2, \cdots, A_\alpha^l)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. It can be shown that if $u_1, u_2, \cdots, u_l \in W^{m,p}(D)$, then

$$A_\alpha(x, \delta u_1, \cdots, \delta u_l, D^m u_1, \cdots, D^m u_l) \in L^{p'}(D) \times \cdots \times L^{p'}(D).$$

Therefore, $\forall u = (u_1, u_2, \cdots, u_l), w = (w_1, w_2, \cdots, w_l) \in W^{m,p}(D) \times W^{m,p}(D) \times \cdots \times W^{m,p}(D)$, we define operator

$$Q(u, w) = \sum_{|\alpha| \le m} \int_D \Big(A_\alpha^1(x, \delta u_1, \cdots, \delta u_l, D^m u_1, \cdots, D^m u_l) D^\alpha w_1 + \cdots$$
$$+ A_\alpha^l(x, \delta u_1, \cdots, \delta u_l, D^m u_1, \cdots, D^m u_l) D^\alpha w_l\Big) dx.$$

For a close subspace V of the interior of $W^{m,p}(D) \times W^{m,p}(D) \times \cdots \times W^{m,p}(D)$, the mapping $w \to Q(u, w)$ is linear and continuous in V. This mapping defines an operator $A(u) \in V'$, the dual space of V, in the following way:

$$Q(u, w) = (A(u), w), \quad \forall w \in V.$$

Therefore, for $u \in \mathcal{D}(\mathcal{D}) \times \cdots \times \mathcal{D}(\mathcal{D})$, vector operator $A(u)$ can be expressed as

$$A(u) = \sum_{|\alpha| \le m} \Big((-1)^{|\alpha|} D^{|\alpha|} A_\alpha^1(x, \delta u_1, \cdots, \delta u_l, D^m u_1, \cdots, D^m u_l), \cdots,$$

$$(-1)^{|\alpha|} D^{|\alpha|} A_\alpha^l(x, \delta u_1, \cdots, \delta u_l, D^m u_1, \cdots, D^m u_l)\Big).$$

Theorem 2.1. *In addition to the above assumptions on* $A_\alpha^1, A_\alpha^2, \cdots, A_\alpha^l$, *we further assume that, for any* $u \in V$,

$$\frac{Q(u, u)}{\| u \|_V} \to \infty, \text{ as } \| u \|_V \to \infty; \tag{2.1}$$

for almost all $x \in D$ *and bounded* η,

$$\frac{\sum_{|\alpha| \le m} A_\alpha^1(x, \eta, \xi)\xi_{1\alpha} + \cdots + A_\alpha^l(x, \eta, \xi)\xi_{l\alpha}}{|\xi_1| + \cdots + |\xi_l| + |\xi_1|^{p-1} + \cdots + |\xi_l|^{p-1}} \to \infty \text{ as } |\xi_1|, \cdots, |\xi_l| \to \infty; \tag{2.2}$$

and for all η *and* $\xi_1 \neq \xi_1^*, \cdots, \xi_l \neq \xi_l^*$,

$$\sum_{|\alpha|=m} (A_\alpha^1(x, \eta, \xi) - A_\alpha^1(x, \eta, \xi^*))(\xi_{1\alpha} - \xi_{1\alpha}^*) + \cdots + \sum_{|\alpha|=m} (A_\alpha^1(x, \eta, \xi)$$

$$- A_\alpha^1(x, \eta, \xi^*))(\xi_{1\alpha} - \xi_{1\alpha}^*) > 0. \tag{2.3}$$

Then, for any $f \in V$, *there exists* $u \in V$ *such that* $A(u) = f$.

Proof. This result is a straight forward generation of Theorem 2.8 in [6] (page 182). □

3. Existence Results

Rewrite the system (1.1) as

$$\frac{\partial c_1}{\partial t} - D_1 \nabla c_1 + \frac{\partial}{\partial x}(uc_1) + \frac{\partial}{\partial y}(vc_1) - (u_x + v_y - \frac{v_s}{h})c_1 - \frac{v_r}{h_1} c_2$$

$$+ c_g \alpha_{pc} c_3 - c_d \alpha_{pc} c_3 = f,$$

$$\frac{\partial c_2}{\partial t} - D_2 \nabla c_2 - \frac{v_s}{h} c_1 + (\frac{v_b}{h_1} + \frac{v_r}{h_1})c_2 - \frac{v_0}{h_1} \alpha_{pc} c_3 = 0,$$

$$\frac{\partial c_3 \alpha_{pc}}{\partial t} - c_g \alpha_{pc} c_3 + c_d \alpha_{pc} c_3 + \frac{v_0}{h_1} \alpha_{pc} c_3 = 0. \tag{3.1}$$

Denote $u = (c_1, c_2, c_3 \alpha_{pc})$. System (3.1) can be expressed as

$$u_t + A_1 u + A_2 u + A_3 u = F, \tag{3.2}$$

where

$$A_1 = \begin{pmatrix} -D_1 \nabla & 0 & 0 \\ 0 & -D_2 \nabla & 0 \\ 0 & 0 & -c_g \end{pmatrix}, \; A_2 = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \zeta_1 + \begin{pmatrix} \partial_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \zeta_2,$$

$$A_3 = \begin{pmatrix} \frac{v_s}{h} - u_x - u_y & \frac{v_r + v_b}{h_1} & c_g - c_d \\ -\frac{v_s}{h} & \frac{v_b}{h_1} + \frac{v_r}{h_1} & \frac{v_0}{h_1} \\ 0 & 0 & c_d + \frac{v_0}{h_1} \end{pmatrix}, \; F = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix},$$

$$\zeta_1 = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \zeta_2 = \begin{pmatrix} v & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $w = (w_1, w_2, w_3)^T$, define a bilinear functional:

$$Q(u, w) = \int_Q \Big((D_1 \frac{\partial c_1}{\partial x} - u c_1) \frac{\partial w_1}{\partial x} + (D_1 \frac{\partial c_1}{\partial y} - v c_1) \frac{\partial w_1}{\partial y} + D_2 \frac{\partial c_2}{\partial x} \frac{\partial w_2}{\partial x}$$

$$+ D_2 \frac{\partial c_2}{\partial y} \frac{\partial w_2}{\partial y} + [(\frac{v_s}{h} - u_x - u_y) c_1 - \frac{v_r}{h_1} c_2 + (c_g - c_d) \alpha_{pc} c_3] w_1$$

$$+ [-\frac{v_s}{h} c_1 + (\frac{v_b}{h_1} + \frac{v_r}{h_1}) c_2 - \frac{v_s}{h_1} c_3] w_2$$

$$+ (-c_g + c_d + \frac{v_0}{h_1}) \alpha_{pc} c_3 w_3 \Big) dx\, dy\, dt. \tag{3.3}$$

We need the following theorem from [6]:

Theorem 3.1. *Suppose that V is a reflexive Banach space and that it is strictly convex with respect to a norm and its dual space V' is strictly convex with respect to the dual norm. Denote L as a maximal monotone linear operator: $D(L) \subset V \to V'$ and Λ as a psuedo-operator: $V \to V$, such that $\frac{(Lmu, u)}{\| u \|_V} \to \infty$ as $\| u \|_V \to \infty$. Then $\forall f \in V'$, there exits $u \in D(L)$, such that $Lu + \Lambda u = f$.*

Take $L = (\frac{\partial}{\partial t}, \frac{\partial}{\partial t}, \frac{\partial}{\partial t})^T$ and $E = \{v : v \in H^1(Q), v|_{\Sigma} = 0\}$. Let $V = E \times E \times L^2(Q)$. For $u = (c_1, c_2, c_3 \alpha_{pc})$, define

$$D(L^*) = \{u \in V : u_t \in V', u|_{t=T} = 0, c_3|_{\Sigma} = 0\}.$$

It is easy to see that $L \geq 0$ (i.e.$(Lu, u) \geq 0$) and that $L^* \geq 0((L^* u, u) \geq 0)$. It can also be shown that L, L^* are maximal monotone operators. Define operator A as follows:

$$(A(u), w) = Q(u, w),$$

where the bilinear functional Q is defined in (3.3). Similarly, with $c_g(|c_1|, |c_2|)$ replacing $c_g(c_1, c_2)$ in (3.3), we can define a bilinear functional $\widetilde{Q}(u, w)$ and an operator B can be defined as

$$(B(u), w) = \widetilde{Q}(u, w).$$

We now check the conditions in Theorem 2.1 for B. For $u = (c_1, c_2, c_3)$, $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$, we decompose $(B(u), v) = B_1(u, v) + B_2(v)$, where

$$(B_1(u, v), w) = \int_Q ((D_1 \frac{\partial c_1}{\partial x} - u_0 v_1) \frac{\partial w_1}{\partial x} + (D_1 \frac{\partial c_1}{\partial y} - v_0 v_2) \frac{\partial w_1}{\partial y}$$

$$+ D_2 \frac{\partial c_2}{\partial x} \frac{\partial w_2}{\partial x} + D_2 \frac{\partial c_2}{\partial y} \frac{\partial w_2}{\partial y}) dx\, dy\, dt,$$

$$(B_2(u),\, w) = \int_Q [(\frac{v_s}{h} - u_{0x} - u_{0y})c_1 - \frac{v_r}{h_1}c_2 + (c_g(|c_1|,\, |c_2|) - c_d)\alpha_{pc}c_3]w_1$$

$$+ [-\frac{v_s}{h}c_1 + (\frac{v_b}{h_1} + \frac{v_r}{h_1})c_2 - \frac{v_s}{h_1}c_3]w_2$$

$$+ (-c_g(|c_1|,\, |c_2|) + c_d + \frac{v_0}{h_1})\alpha_{pc}c_3w_3\Big)dx\, dy\, dt.$$

It is obvious that $B(u) = B(u,\, u)$. Let us check (2.1).

$$(B(u),\, u) = \int_Q \Big(D_1((\frac{\partial c_1}{\partial x})^2 + (\frac{\partial c_1}{\partial y})^2) + D_2((\frac{\partial c_2}{\partial x})^2 + (\frac{\partial c_2}{\partial y})^2)$$

$$+ [(\frac{v_s}{h} - u_{0x} - u_{0y})c_1^2 + (\frac{v_b}{h_1} + \frac{v_r}{h_1})c_2^2(c_d - c_g(|c_1|,\, |c_2|)$$

$$+ \frac{v_0}{h_1})\alpha_{pc}c_3^2 - \frac{u_0}{2}(\frac{\partial c_1}{\partial x})^2 - \frac{v_0}{2}(\frac{\partial c_1}{\partial y})^2 - \frac{v_r}{h_1}c_1c_2$$

$$+ (c_g(|c_1|,\, |c_2|) - c_d)\alpha_{pc}c_1c_3 - \frac{v_s}{h}c_1c_2 - \frac{v_0}{h_1}c_2c_3\Big)dx\, dy\, dt$$

$$\geq \int_Q \Big(D_1((\frac{\partial c_1}{\partial x})^2 + (\frac{\partial c_1}{\partial y})^2) + D_2((\frac{\partial c_2}{\partial x})^2 + (\frac{\partial c_2}{\partial y})^2)$$

$$+ [(\frac{v_s}{2h} + \frac{c_g}{2} - \frac{v_r}{2h_1} - \frac{u_{0x}}{2} - \frac{u_{0y}}{2} - \frac{c_d}{2})c_1^2 + (\frac{v_b}{h_1} + \frac{v_r}{2h_1} - \frac{v_s}{2h}$$

$$- \frac{v_0}{2h_1})c_2^2 + (\frac{1}{2}(c_d - c_g(|c_1|,\, |c_2|)) + \frac{v_0}{2h_1})\alpha_{pc}c_3^2\Big)dx\ dy\, dt.$$

Since $c_g = \nu_m \dfrac{|c_1|}{k|c_2| + |c_1|} \geq \nu_m > 0$, if

$$\frac{v_s}{h} - \frac{v_r}{h_1} - u_{0x} - u_{0y} - c_d \geq 0, \tag{3.4}$$

$$\frac{2v_b}{h_1} + \frac{v_r}{h_1} - \frac{v_s}{h} - \frac{v_0}{h_1} \geq 0, \tag{3.5}$$

$$\frac{1}{2}\left(c_d - c_g(|c_1|, |c_2|)\right) + \frac{v_0}{2h_1} \geq 0, \tag{3.6}$$

there exists a constant c such that

$$(B(u), u) \geq c\| u \|_V^2. \tag{3.7}$$

Condition (2.1) is then satisfied.

Next, we have

$$\frac{(D_1\xi_{11} - u_0\eta_1)\xi_{11} + (D_1\xi_{12} - v_0\eta_1)\xi_{12} + D_2\xi_{21}^2 + D_2\xi_{22}^2}{2[(\xi_{11}^2 + \xi_{12}^2)^{\frac{1}{2}} + (\xi_{21}^2 + \xi_{22}^2)^{\frac{1}{2}}]}$$

$$\geq \frac{D_1(\xi_{11}^2 + \xi_{12}^2) + D_2(\xi_{21}^2 + \xi_{22}^2) - \frac{1}{2D_1}u_0^2\eta_1^2 - \frac{1}{2D_2}v_0^2\eta_1^2}{2[(\xi_{11}^2 + \xi_{12}^2)^{\frac{1}{2}} + (\xi_{21}^2 + \xi_{22}^2)^{\frac{1}{2}}]}. \tag{3.8}$$

For bounded u_0, v_0, and $\eta = (\eta_1, \eta_2)$ in a compact subset of R^2, it is easy to see that as, $|\xi_1| = (\xi_{11}^2 + \xi_{12}^2)^{\frac{1}{2}} \to \infty$, $|\xi_2| = (\xi_{21}^2 + \xi_{22}^2)^{\frac{1}{2}} \to \infty$, expression (3.8) approaches infinity. Therefore, condition (2.2) is satisfied.

Condition (2.3) is obviously satisfied.

Hence, it results from Theorem 2.1.

Theorem 3.2. *Suppose that the velocity of water flow is small along x-axis and y-axis and that conditions* (3.4), (3.5), *and* (3.6) *are satisfied, for $f = (f_1, f_2, f_3) \in V'$, there exists, a unique solution of the initial-boundary value problem*

$$B(u) = f, \tag{3.9}$$

$$u_{t=0} = (c_{10}, c_{20}, c_{30}), \tag{3.10}$$

$$c_1 = c_2 = 0, \quad \textit{when } P(x, y, t) = 0. \tag{3.11}$$

Since c_{10}, c_{20}, c_{30} denote initial phosphorus densities and are therefore nonnegative, we can show that the solution (c_1, c_2, c_3) is positive. For this, we need the following lemma:

Lemma 3.1 (Comparison Lemma [6]). *Given m uniformly parabolic operators*

$$L_i = \frac{\partial}{\partial t} - \sum_{k, j=1}^{n} a_{kj}^{(i)}(x, t) \frac{\partial^2}{\partial x_k \partial x_j} + \sum_{k=1}^{n} b_k^{(i)}(x, t) \frac{\partial}{\partial x_k}, \ m = 1, 2, \cdots, m,$$

and a matrix $(h_{ij})_{m \times m}$ such that $h_{ij} \leq 0$ for $i \neq j$, $i, j = 1, 2, \cdots, m$, if, for $u_i \in C(\overline{Q})$, $i = 1, 2, \cdots, m$

1. $L_i u_i + \sum_{j=1}^{n} < 0$ *(or* ¿ 0*)*, $i = 1, 2, \cdots, m$;

2. $u(x, o) = (u_1(x, 0), u_2(x, 0), \cdots, u_m(x, 0)) < 0$ *(or* ¿ 0*)* *for* $x \in |O$ $M_0 = Q \cap \{t = 0\}$; *and*

3. $u|_{\Sigma} < 0$ *(or* ¿ 0*)*, *then* $u(x, t) < 0$ *(or* ¿ 0*)*, *for* $(x, t) \in Q$.

Theorem 3.3. *Under the assumptions in Theorem* 3.2, *the solution of the initial-boundary value problem*

$$B(u) = f, \tag{3.12}$$

$$u_{t=0} = (c_{10}, c_{20}, c_{30}), \tag{3.13}$$

$$c_1 = c_2 = 0, \quad when\ P(x, y, t) = 0, \tag{3.14}$$

is nonnegative, $u \geq 0$.

Proof. Let $v_1 = c_1 + \frac{\varepsilon}{2} e^{Bt}$, $v_2 = c_2 + \varepsilon e^{Bt}$, $v_3 = c_3$. Then $v = (v_1, v_2, v_3)$ satisfies

$$\frac{\partial v_1}{\partial t} - D_1 \nabla v_1 + u_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + \frac{v_s}{h} v_1 - \frac{v_r}{h_1} v_2$$

$$= f + \left(\frac{v_s}{2h} - \frac{v_r}{h}\right) \varepsilon e^{Bt} + \frac{\varepsilon B}{2} e^{Bt} - (c_g(|c_1|, |c_2|) - c_d) \alpha_{pc} c_3, \tag{3.15}$$

$$\frac{\partial v_2}{\partial t} - D_2 \nabla v_2 + \left(\frac{v_b}{h_1} - \frac{v_r}{h_1} \right) v_2 - \frac{v_s}{h} v_1$$

$$= \left(\frac{v_b}{h_1} + \frac{v_r}{h_1} - \frac{v_s}{2h} \right) \varepsilon e^{Bt} + \frac{v_0}{h_1} \alpha_{pc} c_3 + \varepsilon B e^{Bt}, \tag{3.16}$$

$$\frac{\partial c_3 \alpha_{pc}}{\partial t} = \left(c_g(|c_1|, |c_2|) - c_d - \frac{v_0}{h_1} \right) \alpha_{pc} c_3. \tag{3.17}$$

Since $c_{30} \geq 0$, (3.17) implies that $c_3 \geq 0$. For large enough B, we have from conditions (3.4), (3.5) that

$$\frac{\partial v_1}{\partial t} - D_1 \nabla v_1 + u_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial v_1}{\partial y} + \frac{v_s}{h} v_1 - \frac{v_r}{h_1} v_2 > 0,$$

$$\frac{\partial v_2}{\partial t} - D_2 \nabla v_2 + \left(\frac{v_b}{h_1} + \frac{v_r}{h_1} \right) v_2 - \frac{v_s}{h} v_1 > 0. \tag{3.18}$$

It then results from Lemma 3.1 that

$$v_1 = c_1 + \frac{\varepsilon}{2} e^{Bt} > 0, \; v_2 = c_2 + \varepsilon e^{Bt} > 0.$$

Sending $\varepsilon \to 0$, we have also $c_1 \geq 0$, $c_2 \geq 0$. □

Finally, we have

Theorem 3.4. *Suppose that the velocity of water flow is small along x-axis and y-axis and that conditions* (3.4), (3.5), *and* (3.6) *are satisfied, for $f = (f_1, f_2, f_3) \in V'$, there exists, a unique solution of the initial-boundary value problem*

$$A(u) = f, \tag{3.19}$$

$$u_{t=0} = (c_{10}, c_{20}, c_{30}), \tag{3.20}$$

$$c_1 = c_2 = 0, \quad \textit{when } P(x, y, t) = 0. \tag{3.21}$$

Proof. Since the solution obtained in Theorem 3.2 is positive, $c_g(|c_1|, |c_2|) = c_g(|c_1|, |c_2|)$. Therefore, $A(u) = B(u)$. □

Remark 1. In reality, condition (3.4) is satisfied, when the rate at which phosphate sinks is greater than the rate at which the sediment releases phosphate, and when the speed of the water flow and the rate at which the organisms die are slow.

Remark 2. Condition (3.5) is satisfied, when the rate at which the top layer of lake bottom gets buried by new dirt in the water and by newly died micro-plants is relatively large.

Remark 3. Condition (3.6) is satisfied, when the depth of the sediment is small.

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